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Stability of Sets for Impulsive Systems

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Problems related to the stability and asymptotic stability of sets of sufficiently general type with respect to impulsive systems are considered. The research is done by means of piecewise continuous anxiliary functions which are an analogue of the classical Lyapunov functions. It is proved that the existence of such functions with certain properties is a sufficient condition for various types of stability and asymptotic stability of sets with respect to impulsive systems.

1. INTRODUCTION

Systems of impulsive differential equations are an adequate mathematical model for many problems of physics, technology, biology, etc. The study of various actual processes subject to short-time perturbations during their evolution leads to the necessity of considering such systems which have recently been the object of intensive research (Dishliev and Bainov, 1984, 1985; Leela, 1977; Mil'man and Myshkis, 1960; Pandit, 1977; Pavlidis, 1967; Rama Mohana Rao and Sree Hari Rao, 1977; Samoilenko and Perestyuk, 1981; Simeonov and Bainov, 1985*a*,*b*, 1986).

In the present paper some problems related to stability and asymptotic stability of sets of sufficiently general type contained in some domain (an open connected set) are considered. The research is done by means of piecewise continuous anxiliary functions which are an analogue of Lyapunov functions. Sufficient conditions for stability and asymptotical stability of sets are obtained which are a generalization of the results obtained by Lyapunov, Persidskii, Massera, Salvadori, and others concerning the stability of the stationary solution of systems of ordinary differential equations without impulses.

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Results related to the study of the stability of sets for systems of differential equations without impulses have been obtained by Hale and Stokes, (1960), Lefschetz (1958), Rouche *et al.*, (1977), and Yoshizawa (1959, 1962).

2. PRELIMINARIES

Denote by $\mathbb{R}^n$ an n-dimensional Euclidean space with a norm $\|\cdot\|$ and scalar product $\langle\ ,\ \rangle$. Let $I=[0,\infty)$ and Ω be a domain in $\mathbb{R}^n$.

Consider the following system of impulsive differential equations:

$$\begin{aligned}\frac{dx}{dt}&=f(t,x), \qquad t\neq\tau_i(x);\\ \Delta x|_{t=\tau_i(x)}&=I_i(x)\end{aligned}\tag{1}$$

where

$$x\in\mathbb{R}^n, f\colon I\times\Omega\to\mathbb{R}^n,\ I_i\colon\Omega\to\mathbb{R}^n,\ \tau_i\colon\Omega\to\mathbb{R},$$

and

$$\Delta x|_{t=\tau_i(x)}=x(t+0)-x(t-0), \qquad i=1,2,\ldots$$

Such systems are characterized by the fact that under the momentary action of a force (impact, impulse) by meeting some of the hypersurfaces $\sigma_i=\{(t,x)\in I\times\Omega\colon t=\tau_i(x)\}$ the mapping point $(t,x(t))$ of the extended phase space $I\times\Omega$ is transferred instantly from the position (t,x) to the position $(t,x+I_i(x))$. The solutions $x(t)$ of system (1) are piecewise continuous functions with points of discontinuity of the first type at which they are left continuous, i.e., at the moment t_i when the integral curve $(t,x(t))$ meets the hypersurface σ_i the following relations are fulfilled:

$$\begin{aligned}&x(t_i-0)=x(t_i),\\ &\Delta x|_{t=t_i}=x(t_i+0)-x(t_i-0)=I_i(x(t_i)), \qquad i=1,2,\ldots\end{aligned}$$

Assume that $M\subset I\times\Omega$. Introduce the notations

$$M(t)=\{x\in\Omega\colon (t,x)\in M\}, \qquad t\in I, \qquad B_\alpha=\{x\in\mathbb{R}^n\colon \|x\|<\alpha\},$$
$$\bar{B}_\alpha=\{x\in\mathbb{R}^n\colon \|x\|\le\alpha\}, \qquad \alpha>0$$

and denote by $M(t,\varepsilon)$ the ε-neighborhood of $M(t)$ in Ω, i.e., $M(t,\varepsilon)=\{x\in\Omega\colon d(x,M(t))<\varepsilon\}$, where $d(x,M(t))=\inf_{y\in M(t)}\|x-y\|$ is the distance between x and $M(t)$.

Assume that $t_0\in I, x_0\in\Omega$. Denote by $x(t;t_0,x_0)$ the solution of system (1) which satisfies the initial condition $x(t_0+0;t_0,x_0)=x_0$ and by $J^+ = J^+(t_0, x_0)$ the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, x_0$ is defined.

We shall say that conditions (A) are satisfied if the following conditions hold:

A1. The function $f(t, x)$ is continuous in $I \times \Omega$ and satisfies the Lipschitz condition with respect to $x \in \Omega$ with a constant $L > 0$.

A2. $\|f(t, x)\| \leq N < \infty$ for $t \in I, x \in \Omega$ $(N \geq 0)$.

A3. $\|I_i(x) - I_i(y)\| \leq c\|x - y\|$ for $x, y \in \Omega$, $i = 1, 2, \ldots$.

A4. For all $x \in \Omega$, $x + I_i(x) \in \Omega$ $(i = 1, 2, \ldots)$.

A5. The functions $\tau_i(x)$, $i = 1, 2, \ldots$, are continuous in Ω, $0 < \tau_i(x) < \tau_2(x) < \cdots$, $\lim_{i \to \infty} \tau_i(x) = \infty$ uniformly on $x \in \Omega$ and $\inf_\Omega \tau_{i+1}(x) - \sup_\Omega \tau_i(x) \geq \theta > 0$.

We shall assume that for system (1) the phenomenon of "heating" is absent, i.e., that the following condition (B) is fulfilled:

B. The integral curve of each solution of system (1) meets each hypersurface σ_i at most once.

Effective sufficient conditions for the absence of the phenomenon of "heating" are given by Dishliev and Bainov (1984, 1985).

We shall say that conditions (C) are satisfied if the following conditions hold:

C1. For each $t \in I$ the set $M(t)$ is nonempty.

C2. There exists a compact set $Q \subset \Omega$ such that $M(t) \subset Q$ for all $t \in I$.

C3. For any compact subset F of $I \times \Omega$ there exists a constant $K > 0$ dependent on F such that if (t, x), $(t', x) \in F$, then the following inequality holds:

$$|d(x, M(t)) - d(x, M(t')| \leq K|t - t'|$$

We shall say that condition (D) is satisfied if the following condition holds:

D. Each solution $x(t; t_0, x_0)$ of system (1) satisfying the estimate

$$d(x(t; t_0, x_0), M(t)) \leq h \qquad (\{x \in \mathbb{R}^n \colon d(x, M(t)) \leq h\} \subset \Omega)$$

for $t \in J^+(t_0, x_0)$ is defined in the interval (t_0, ∞).

We shall give some definitions of stability and attraction of sets for system (1) which correspond to the definitions in Samoilenko and Perestyluk (1981) and have the form used in Rouche *et al.* (1977).

Definition 1. The set M is called:

(a) A stable set of system (1) if

$$(\forall \varepsilon > 0)(\forall \alpha > 0)(\forall t_0 \in I)[\exists \delta = \delta(t_0, \alpha, \varepsilon) > 0][\forall x_0 \in \bar{B}_\alpha \cap M(t_0, \delta)]$$
$$[\forall t \in J^+(t_0, x_0)]\colon \quad x(t; t_0, x_0) \in M(t, \varepsilon)$$

(b) A t-uniformly stable set of system (1) [or α-uniformly stable) set of system (1)] if the number δ from (a) does not depend on t_0 (or on α).

(c) A uniformly stable set of system (1) if the number δ from (a) depends only on ε.

Definition 2. The set M is called:

(a) An attractive set of system (1) if

$$(\forall\alpha>0)(\forall t_0\in I)(\exists\lambda>0)(\forall\varepsilon>0)[\forall x_0\in\bar{B}_\alpha\cap M(t_0,\lambda)]$$
$$[\exists\sigma>0\colon t_0+\sigma\in J^+(t_0,x_0)][\forall t\geq t_0+\sigma,\ t\in J^+(t_0,x_0)]\colon$$
$$x(t;\,t_0,x_0)\in M(t,\varepsilon)$$

(b) A t-uniformly attractive set of system (1) if

$$(\forall\alpha>0)(\exists\lambda>0)(\forall\varepsilon>0)(\exists\sigma>0)(\forall t_0\in I)$$
$$[\forall x_0\in\bar{B}_\alpha\cap M(t_0,\lambda)\colon\ t_0+\sigma\in J^+(t_0,x_0)]$$
$$(\forall t\geq t_0+\sigma,\ t\in J^+)\colon\quad x(t;\,t_0,x_0)\in M(t,\varepsilon)$$

(c) An α-uniformly attractive set of system (1) if

$$(\forall t_0\in I)(\exists\lambda>0)(\forall\varepsilon>0)(\exists\sigma>0)(\forall\alpha>0)$$
$$[\forall x_0\in\bar{B}_\alpha\cap M(t_0,\lambda)\colon\ t_0+\sigma\in J^+(t_0,x_0)]$$
$$(\forall t\geq t_0+\sigma,\ t\in J^+)\colon\quad x(t;\,t_0,x_0)\in M(t,\varepsilon)$$

(d) A uniformly attractive set of system (1) if

$$(\exists\lambda>0)(\forall\varepsilon>0)(\exists\sigma>0)(\forall\alpha>0)(\forall t_0\in I)$$
$$[\forall x_0\in\bar{B}_\alpha\cap M(t_0,\lambda)\colon\ t_0+\sigma\in J^+(t_0,x_0)]$$
$$(\forall t\geq t_0+\sigma,\ t\in J^+)\colon\quad x(t;\,t_0,x_0)\in M(t,\varepsilon)$$

Definition 3. The set M is called:

(a) An asymptotically attractive.

(b) A t-uniformly (or α-uniformly) asymptotically stable set of system (1) if it is t-uniformly (or α-uniformly) stable and t-uniformly (or α-uniformly) attractive.

(c) A uniformly asymptotically stable set of sysem (1) if it is uniformly stable and uniformly attractive.

Remark 1. If condition C2 holds, then the numbers δ from Definition 1a and λ and σ from Definition 2a can be chosen independent of α. Therefore, if the set M is stable, then it is α-uniformly stable and if the set M is attractive, then it is α-uniformly attractive.

In the following considerations we shall use a class $\mathcal{V}_0$ of piecewise continuous auxiliary functions $V; I\times\Omega\to\mathbb{R}$ which are an analogue of Lyapunov functions (Simeonov and Batinov, 1986).

Put $\tau_0(x)\equiv 0$ for $x\in\Omega$. Consider the sets

$$G_i=\{(t,x)\in I\times\Omega\colon \tau_{i-1}(x)<t<\tau_i(x)\},\qquad i=1,2,\ldots$$

$$G=\bigcup_1^\infty G_i$$

Definition 4. We shall say that the function $V\colon I\times\Omega\to\mathbb{R}$ belongs to the class $\mathcal{V}_0$ if the following conditions hold:

1. V is continuous in G and is locally Lipschitz continuous on x in any one of the sets G_i.
2. $V(t,x)=0$ for $(t,x)\in M$ and $V(t,x)>0$ for $(t,x)\in(I\times\Omega)\backslash M$.
3. For any $i=1,2,\ldots$ and each point $(\xi,\eta)\in\sigma_i$ there exists the following finite limits:

$$V(\xi-0,\eta)=\lim_{\substack{(t,x)\to(\xi,\eta)\\ (t,x)\in G_i}} V(t,x),\qquad V(\xi+0,\eta)=\lim_{\substack{(t,x)\to(\xi,\eta)\\ (t,x)\in G_{i+1}}} V(t,x)$$

and the equality $V(\xi-0,\eta)=V(\xi,\eta)$ holds.

4. For each point $(t,x)\in\sigma_i=1,2,\ldots$, the following inequality holds:

$$V(t+0,x+I_i(x))\le V(t,x) \tag{2}$$

Assume that $V\in\mathcal{V}_0$. For $(t,x)\in G$ put

$$\dot V_{(1)}(t,x)=\lim_{h\to 0^+}\sup h^{-1}[V(t+h,x+hf(t,x))-V(t,x)]$$

Note that if $x=x(t)$ is a solution of system (1), then for $(t,x)\in G$ [i.e., $t\ne\tau_i(x)$] the equality $\dot V_{(1)}(t,x)=D^+V(t,x)$ is satisfied, where

$$D^+V(t,x)=\lim_{h\to 0^+}\sup h^{-1}[V(t+h,x(t+h))-V(t,x(t))]$$

is the upper right Dini derivative of the function $V(t,x(t))$.

Moreover, if the function V belongs to class $\mathcal{V}_0$ and satisfies the condition $\dot V_{(1)}(t,x)\le 0$ for $(t,x)\in G$ and if $x(t;t_0,x_0)$ is a solution of system (1), then the function $V(t,x(t;t_0,x_0))$ is monotonely decreasing in the interval $J^+(t_0,x_0)$.

Denote by $\mathcal{K}$ the class of all continuous and strictly increasing functions $a\colon I\to I$ such that $a(0)=0$.

3. MAIN RESULTS

We shall find sufficient conditions for stability and asymptotic stability of the set M with respect to system (1) in which functions of class $\mathcal{V}_0$ are used.

Theorem 1. Let conditions (A), (B), C1, C3, and (D) hold and functions $V \in \mathscr{V}_0$ and $a \in \mathscr{K}$ exist such that the following relations are satisfied:

$$a(d(x, M(t))) \leq V(t, x) \qquad \text{for} \quad (t, x) \in I \times \Omega \tag{3}$$

$$\dot{V}_{(1)}(t, x) \leq 0 \qquad \text{for} \quad (t, x) \in G \tag{4}$$

Then the set M is a stable set of system (1).

Proof. Assume that $\varepsilon > 0, \alpha > 0$, and $t_0 \in I$. From the condition $V(t_0, x) = 0$ for $x \in M(t_0)$ it follows that there exists a number $\delta = \delta(t_0, \alpha, \varepsilon) > 0$ such that if $x \in \bar{B}\alpha \cap M(t_0, \delta) \cap \Omega$, then $V(t_{0+}, 0, x) < a(\varepsilon)$.

Let $x_0 \in \bar{B}_\alpha \cap M(t_0, \delta) \cap \Omega$. From (2) and (4) it follows that the function $V(t, x(t; t_0, x_0))$ is decreasing in $J^+(t_0, x_0)$. Using this fact and (3), we obtain

$$a(d(x(t; t_0, x_0), M(t))) \leq V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) < a(\varepsilon)$$

for $t \in J^+(t_0, x_0)$, hence $J^+(t_0, x_0) = (t_0, \infty)$ and $x(t; t_0, x_0) \in M(t, \varepsilon)$ for all $t > t_0$.

Theorem 2. Let the conditions of Theorem 1 hold and a function $b\colon r \to b(t, r) \in \mathscr{K}$ for each $t \in I$ fixed exist such that the following inequality holds:

$$V(t, x) \leq b(t, d(x, M(t))) \qquad \text{for} \quad (t, x) \in I \times \Omega \tag{5}$$

Then the set M is an α-uniformly stable set of system (1).

Proof. Assume that $\varepsilon > 0$ and $t_0 \in I$. Choose a number $\delta = \delta(t_0, \varepsilon) > 0$ such that $b(t_0, \delta) < a(\varepsilon)$. Let $\alpha > 0$ and $x_0 \in \bar{B}_\alpha \cap M(t_0, \delta) \cap \Omega$. Using successively (3), (4), (2), and (5), we obtain

$$\begin{aligned} &a(d(x(t; t_0, x_0), M(t))) \\ &\quad \leqq V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \\ &\quad \leq b(t_0, d(x_0, M(t_0))) \leq b(t_0, \delta) < a(\varepsilon) \end{aligned}$$

for $t \in J^+(t_0, x_0)$. Hence $J^+(t_0, x_0) = (t_0, \infty)$ and $x(t; t_0, x_0) \in M(t, \varepsilon)$ for $t > t_0$. In the same way the following two theorems are proved.

Theorem 3. Let the conditions of Theorem 1 hold and a function $b\colon r \to (r, s) \in \mathscr{K}$ for each $s \geqq 0$ fixed exist such that the following inequality is satisfied:

$$V(t, x) \leq V(d(x, M(t)), \|x\|), \qquad (t, x) \in I \times \Omega$$

Then the set M is a t-uniformly stable set of system (1).

Theorem 4. Let the conditions of Theorem 1 hold and a function $b \in \mathscr{K}$ exist such that

$$V(t, x) \leq b(d(x, M(t))), \qquad (t, x) \in I \times \Omega$$

Then the set M is a uniformly stable set of system (1).

In the following theorem two auxiliary functions of class $\mathscr{V}_0$ are used.

Theorem 5. Let conditions (A), (B), C1, C3, and (D) hold and functions V, $W \in \mathscr{V}_0$ and a, b, $c \in \mathscr{K}$ exist such that

$$a(d(x, M(t))) \leq V(t, x), \qquad (t, x) \in I \times \Omega \tag{6}$$

$$b(d(x, M(t))) \leq W(t+0, x), \qquad (t, x) \in I \times \Omega \tag{7}$$

$$\dot{V}_{(1)}(t, x) \leq -c(W(t, x)), \qquad (t, x) \in G \tag{8}$$

$$\sup\{\dot{W}_{(1)}(t, x) \colon (t, x) \in G\} \leq N_1 < \infty \tag{9}$$

Then the set M is an asymptotically stable set of system (1).

Proof. From Theorem 1 it follows that the set M is a stable set of system (1).

Let the number $\alpha > 0$ be such that $M(t, \alpha) \subset \Omega$ for all $t \in I$. For an arbitrary $t \in I$ put

$$V_{t,\alpha}^{-1} = \{x \in \Omega \colon V(t+0, x) \leq a(\alpha)\}$$

From (6) it follows that for all $t \in I$ the inclusion

$$V_{t,\alpha}^{-1} \subset M(t, \alpha) \subset \Omega \tag{10}$$

holds. By (8) and (2) we find that if $t_0 \in I$ and $x_0 \in V_{t_0,\alpha}^{-1}$, then $x(t; t_0, x_0) \in V_{t,\alpha}^{-1}$ for $t \in J^+(t_0, x_0)$ and by (10) we conclude that the solution $x(t) = x(t; t_0, x_0)$ cannot reach the boundary of Ω. Hence $J^+(t_0, x_0) = (t_0, \infty)$.

Assume that $t_0 \in I$ and $x_0 \in V_{t_0,\alpha}^{-1}$. We shall prove that

$$\lim_{t \to \infty} d(x(t; t_0, x_0), M(t))) = 0.$$

Suppose that this is not true. Then there exists a sequence $\{\xi_i\}$ diverging to ∞ for $i \to \infty$ and such that $d(x(\xi_i, t_0, x_0), M(\xi_i))) \geq r$ $(i = 1, 2, \ldots)$ for some positive number r. Then from (7) it follows that

$$W(\xi_i, x(\xi_i, t_0, x_0)) \geq b(r), \qquad i = 1, 2, \ldots \tag{11}$$

Considering a subsequence $\{\xi_i\}$ and using the same notations for its members, we can assume that $\xi_i - \xi_{i-1} \geq \beta > 0$ and $\xi_i > t_0$ for each $i = 1, 2, \ldots$. We choose a positive number γ such that $\gamma < \min\{\beta, b(r)/2N_1\}$ and, by (2), (9), and (11), we obtain

$$W(t, x(t; t_0, x_0)) \geq W(\xi_i, x(\xi_i; t_0, x_0)) + \int_{\xi_i}^{t} (\dot{W}_{(1)}(s, x(s; t_0, x_0)) \, ds$$

$$\geq b(r) - N_1(\xi_i - t) \geq b(r) - N_1\gamma > b(r)/2$$

for $t \in [\xi_i - \gamma, \xi_i]$.

Henceforth, applying (2) and (8), we get

$$\begin{aligned} 0 &\leq V(\xi_k, x(\xi_k; t_0, x_0)) \\ &\leq V(t_0+0, x_0) + \int_{t_0}^{\xi_k} \dot{V}_{(1)}(s, x(s; t_0, x_0))\, ds \\ &\leq V(t_0+0, x_0) - \int_{t_0}^{\xi_k} c(W(s, x(s; t_0, x_0)))\, ds \\ &\leq V(t_0+0, x_0) - \sum_{i=1}^{k} \int_{\xi_i-\gamma}^{\xi_i} (W(s, x(s; t_0, x_0)))\, ds \\ &\leq V(t_0+0, x_0) - c\left(\frac{b(r)}{2}\right)\gamma k \to -\infty \qquad \text{for } k \to \infty \end{aligned}$$

which contradicts (6).

Hence $\lim_{t\to\infty} d(x(t; t_0, x_0), M(t)) = 0$ and since $V_{t_0, \alpha}^{-1}$ is a neighborhood of the origin which is contained in $M(t_0, \alpha)$, then the set M is an attractive set of system (1). Theorem 5 is proved. ■

In Theorem 5 the function $W(t, x)$ may have a special form. In the case when $W(t, x) = d(x, M(t))$, we deduce the following corollary.

Corollary 1. Let the following conditions be satisfied:

1. Conditions (A), (B), C1, C3, and (D) hold.
2. There exist functions $V \in \mathscr{V}_0$ and a, $c \in K$ such that

$$a(d(x, M(t))) \leq V(t, x) \qquad \text{for} \quad (t, x) \in I \times \Omega \tag{12}$$

$$\dot{V}_{(1)}(t, x) \leq -c(d(x, M(t))) \qquad \text{for} \quad (t, x) \in G \tag{13}$$

$$d(x + I_i(x), M(t)) \leq d(x, M(t)), \qquad (t, x) \in \sigma_i \tag{14}$$

3. The function $f(t, x)$ is bounded in the domain $I \times \Omega$.

Then the set M is an asymptotically stable set of system (1).

In fact, without loss of generality we can assume that Ω is a bounded set. Then from the boundedness of $f(t, x)$ it follows that the derivative of $d(x, M(t))$ with respect to system (1) is bounded and the conditions of Theorem 5 are therefore satisfied.

In the case when $W(t, x) = V(t, x)$ we deduce the following corollary.

Corollary 2. Let conditions (A), (B), C1, C3, and (D) hold and functions $V \in \mathscr{V}_0$ and a, $c \in \mathscr{K}$ exist such that

$$a(d(x, M(t)) \leq V(t, x) \qquad \text{for} \quad (t, x) \in I \times \Omega$$

$$\dot{V}_{(1)}(t, x) \leq -c(V(t, x)) \qquad \text{for} \quad (t, x) \in G$$

Then the set M is an asymptotically stable set of system (1).

Theorem 6. Let conditions (A), (B), C1, C3, and (D) hold and functions $V \in \mathscr{V}_0$ and $a, b, c \in \mathscr{K}$ exist such that

$$a(d(x, M(t))) \leq V(t, x) \leq b(d(x, M(t))), \qquad (t, x) \in I \times \Omega \tag{15}$$

$$\dot{V}_{(1)}(t, x) \leq -c(d(x, M(t))), \qquad (t, x) \in G \tag{16}$$

Then the set M is a uniformly asymptotically stable set of system (1).

Proof. From Theorem 4 it follows that the set M is a uniformly stable set of system (1).

Assume that $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > b^{-1}(a(\varepsilon))$. For $t_0 \in I$ and $x_0 \in M(t_0, \delta)$, by (15), (16), and (2), we obtain

$$\begin{aligned} a(d(x(t; t_0, x_0))) &\leq V(t, x(t; t_0, x_0)) \leq V(t_0 + 0, x_0) \\ &\leq b(d(x_0, M(t_0))) \leq b(\delta(\varepsilon)) < a(\varepsilon) \end{aligned}$$

for all $t \in J^+(t_0, x_0)$. Hence, $J^+(t_0, x_0) = (t_0, \infty)$ and $x(t; t_0, x_0) \in M(t, \varepsilon)$ for $t > t_0$.

Let $\lambda = \sup_{\varepsilon \geq 0} \delta(\varepsilon)$, $\sigma = \sigma(\varepsilon) > b(\lambda)/c(\delta(\varepsilon))$, $\alpha > 0$, and $x_0 \in \bar{B}_\alpha \cap M(t_0, \lambda) \cap \Omega$. Assume that for all $t \in [t_0, t_0 + \sigma]$ the inequality $d(x(t; t_0, x_0), M(t))) \geq \delta(\varepsilon)$ holds. Then by (16) we obtain

$$\int_{t_0}^{t} \dot{V}_{(1)}(s, x(s; t_0, x_0))\, ds \leq -c(\delta(\varepsilon)[t - t_0] \tag{17}$$

On the other hand, if $\{t_i\}$ are the moments at which the integral curve of the solution $x(t; t_0, x_0)$ meets respectively the hypersurfaces $\{\sigma_i\}$, then by (2) we obtain

$$\begin{aligned} &\int_{t_0}^{t} \dot{V}_{(1)}(s, x(s; t_0, x_0))\, ds \\ &= \sum_{i=1}^{k} \int_{t_{i-1}+0}^{t_i} \dot{V}_{(1)}(s, x(s; t_0, x_0))\, ds + \int_{t_k+0}^{t-0} \dot{V}_{(1)}(s, x(s; t_0, x_0))\, ds \\ &= \sum_{i=1}^{k} [V(t_i, x(t_i; t_0, x_0)) - V(t_{i-1}+0, x(t_{i-1}+0; t_0, x_0))] \\ &\quad + V(t, x(t; t_0, x_0)) \\ &\quad - V(t_k + 0, x(t_k + 0; t_0, x_0)) \\ &\geq \sum_{i=1}^{k} [V(t_i; x(t_i; t_0, x_0)) - V(t_{i-1}, x(t_{i-1}; t_0, x_0))] \\ &\quad + V(t, x(t; t_0, x_0) - V(t_k, x(t_k; t_0, x_0)) = V(t, x(t; t_0, x_0)) \\ &\quad - V(t_0 + 0, x_0) \end{aligned}$$

for $t \in (t_k, t_{k+1}]$.

From this inequality and (17) we obtain the inequality

$$V(t, x(t; t_0, x_0)) \leq V(t_0+0, x_0) - c(\delta(\varepsilon))[t-t_0]$$

for $t \in [t_0, t_0+\sigma]$, hence for $t = t_0+\sigma$ we have

$$V(t, x(t; t_0, x_0)) \leq V(t_0+0, x_0) - c(\delta)\sigma < b(\lambda) - c(\delta)\frac{b(\lambda)}{c(\delta)} = 0$$

which contradicts (15).

This contradiction shows that there exists a number $t' \in [t_0, t_0+\sigma]$ such that $x(t'; t_0, x_0) \in M(t', \delta)$. Then for $t \geq t'$, hence for all $t \geq t_0+\sigma$ as well, the inequalities

$$\begin{aligned} &a(d(x(t; t_0, x_0), M(t))) \\ &\quad \leq V(t, x(t; t_0, x_0)) \leq V(t', x(t'; t_0, x_0)) \\ &\quad \leq b(d(x(t'; t_0, x_0), M(t'))) \leq b(\delta) < a(\varepsilon) \end{aligned}$$

i.e., $x(t; t_0, x_0) \in M(t, \varepsilon)$, hold for $t \geq t_0+\sigma$, which shows that the set M is a uniformly attractive set of system (1). Theorem 6 is proved. ■

The following theorems are proved analogously.

Theorem 7. Let the following conditions be fulfilled:

1. Conditions (A), (B), (C1, C3, and (D) hold.

2. There exist functions $V \in \mathscr{V}_0$, $a, c \in \mathscr{K}$ and a function $b: r \to b(t, r) \in \mathscr{K}$ for all $t \in I$ fixed such that the following inequalities hold:

$$\begin{aligned} a(d(x, M(t))) &\leq V(t, x) \\ &\leq b(t, d(x, M(t))), \qquad (t, x) \in I \times \Omega \\ \dot{V}_{(1)}(t, x) &\leq -c(d(x, M(t))), \qquad (t, x) \in G \end{aligned}$$

Then the set M is a t-uniformly asymptotically stable set of system (1).

Theorem 8. Let the following conditions be satisfied:

1. Conditions (A), (B), C1, C3, and (D) hold.

2. There exist functions $V \in \mathscr{V}_0$ and $a, c \in \mathscr{K}$ and a function $b: r \to b(r, s) \in \mathscr{K}$ for all $s \geq 0$ fixed such that the following inequalities hold:

$$\begin{aligned} a(d(x, M(t))) &\leq V(t, x) \\ &\leq b(d(x, M(t)), \|x\|), \qquad (t, x) \in I \times \Omega \\ \dot{V}_{(1)}(t, x) &\leq -c(d(x, M(t))), \qquad (t, x) \in G \end{aligned}$$

Then the set M is an α-uniformly asymptotically stable set of system (1).

4. EXAMPLES

Example 1. Consider the system of impulsive differential equations

$$\frac{dx}{dt}=\begin{cases}[B(t)+A(t)]x, & x>0, \quad t\neq t_i\\ 0, & x\leq 0, \quad t\neq t_i\end{cases} \tag{18}$$

$$\Delta x|_{t=t_i}=\begin{cases}I_i(x), & x>0\\ 0, & x\leq 0\end{cases}$$

where $x\in\mathbb{R}^n$, $B(t)$ and $A(t)$ are $(n\times n)$ matrices of continuous functions, $B(t)$ is diagonal and $A(t)$ is skewsymmetric, $I_i(x)$, $i=1,2,\dots,$ are continuous and such that $x+I_i(x)>0$, and $\|x+I_i(x)\|\leq\|x\|$ for $x>0$ ($x>0$ means that $x_i>0$ for $i=1,2,\dots,n$, where x_i is the ith component of the vector $x\in\mathbb{R}^n$). The impulsive moments $\{t_i\}$ form a strictly increasing sequence, i.e., $0<t_1<t_2<\cdots$ and $\lim_{i\to\infty} l_i=\infty$.

Let $M=I\times\{x\in\mathbb{R}^n:x\leq 0\}$. Consider the functions

$$V(t,x)=\begin{cases}\langle x,x\rangle & \text{for} \quad x>0\\ 0 & \text{for} \quad x\leq 0\end{cases}$$

Then

$$\dot V_{(18)}(t,x)=\begin{cases}2\langle x,B(t)x\rangle & \text{for} \quad x>0, \quad t\neq t_i\\ 0 & \text{for} \quad x\leq 0, \quad t\neq t_i\end{cases}$$

Hence, if the elements $b_i(t)$, $i=1,2,\dots,n$, of the matrix $B(t)$ assume nonpositive values for $t\in I$, then $\dot V_{(1)}(t,x)\leq 0$.

Moreover, we have

$$V(t_i+0,x+I_i(x))=\begin{cases}\|x+I_i(x)\| & \text{for} \quad x>0\\ 0 & \text{for} \quad x\leq 0\end{cases}$$

hence $V(t_i+0,x+I_i(x))\leq V(t_i,x)$.

Since $d(x,M(t))=\|x\|$ for all $t\in I$ and $x>0$, then by Theorem 4 the set M is uniformly stable with respect to system (18).

Moreover, if $b_i(t)\leq\gamma_i<0$ for $i=1,2,\dots,n$, then by Theorem 6 the set M is uniformly asymptotically stable with respect to system (18).

Example 2. Consider the system

$$\frac{dx}{dt}=a(t)y+b(t)x(x^2+y^2)$$

$$\frac{dy}{dt}=-a(t)x+b(t)y(x^2+y^2), \qquad t\neq t_i \tag{19}$$

$$\Delta x|_{t=t_i}=c_ix(t_i), \qquad \Delta y|_{t=t_i}=d_iy(t_i)$$

where $x, y \in \mathbb{R}$, the functions $a(t)$ and $b(t)$ are continuous in I, $b(t) \leq 0$, $-1 < c_i \leq 0$, $-1 < d_i \leq 0$, for $i = 1, 2, \ldots, 0 < t_1 < t_2 < \cdots$, and $\lim_{i \to \infty} t_i = \infty$.

Let $M = \{(t, 0, 0): t \in I\}$.

The function $V(t, x, y) = x^2 + y^2$ satisfies the conditions of Theorem 4. In fact,

$$\dot{V}_{(19)}(t, x, y) = 2b(t)(x^2 + y^2)^2 \leq 0 \qquad \text{for} \quad t \neq t_i$$

$$V(t_i + 0, x + c_i x, y + d_i y) = (1 + c_i)^2 x^2 + (1 + d_i)^2 y^2 \leq V(t_i, x, y)$$

Hence the set M is uniformly stable with respect to system (19), i.e. the zero solution of system (19) is uniformly stable [cf., for instance, Simeonov and Bainov (1985*a*), Definition 2].

Moreover, if $b(t) \leq -\gamma < 0$ for $t \in I$, then by Theorem 6 the set M is uniformly asymptotically stable with respect to system (19), i.e., the zero solution of (19) is uniformly asymptotically stable.

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